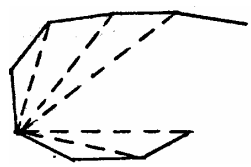
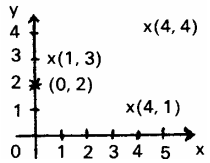
UNIT 7: Coordinate Geometry

Specific Objectives:

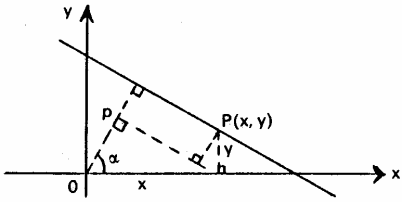
1. To find areas of plane rectilinear figures in the coordinate plane.
2. To find the angle between two straight lines.
3. To understand the normal form of a straight line equation and to apply the knowledge in finding distances.
4. To find the equation of a circle and the points of intersection of a straight line and a circle.
5. To find the equations of tangents to a circle.
6. To find the equations of families of straight lines and circles.
7. To find the equations of a parabola, an ellipse and a hyperbola in standard position.
8. To acquire the concept of parametric equations and locus, and solve simple locus problem.

Detailed Content	Time Ratio	Notes on Teaching
7.1 Plane Rectanglular Coordinates and Distance Between Two Points	1*	Since students should have learnt some elementary concepts of plane rectangular coordinates in S1-3, only a brief revision is necessary at this stage. Distance formula and Section formula could be stated. For Section formula, the external division should be emphasised, i.e. if R is a point on PQ produced such that PR:RQ = r:1, then r will have a negative value. *Example* The centre of a circle is at O(3,4). If A(1,1) is an end point of a diameter, find the coordinates of the other end point.
7.2 Area of Rectilinear Figures	3	Students should be led to discover that the area of a triangle with vertices at O(0,0), $A(x_1,y_1)$ and $B(x_2,y_2)$ may be expressed as the algebraic sum of two right-angled triangles and a trapezium, and is hence given in magnitude by $\frac{1}{2}(x_1y_2 - x_2y_1)$. They should also note that if the vertices O, A and B are chosen in counterclockwise direction then the value obtained will be positive, otherwise a negative value will be obtained.

Detailed Content	Time Ratio	Notes on Teaching
		With the above knowledge in hand, the area of any triangle on the coordinate plane may be taken to be the algebraic sum of the areas of three triangles each with a vertex at the origin, and may hence be calculated by the formula. Area of triangle: $= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3)$ where (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the coordinates of the vertices of the triangle, taken in the counterclockwise direction. Students should be told that for convenience, the formula may be written in the form Area of triangle $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$ where the terms x_1y_2, x_2y_3 and x_3y_1 are regarded as downward products and are positive, and the terms x_2y_1, x_3y_2 and x_1y_3 are regarded as upward products and are negative, and the area is equal to half the sum of these terms. In the final stage, the area of any n-sided polygon on the plane may be calculated by regarding it as composed of (n-2) triangles, as in the following diagram: 

Detailed Content	Time Ratio	Notes on Teaching
		The following formula should be obtained: Area of polygon $=\frac{1}{2}(x_1y_2+x_2y_3+\ldots+x_ny_1-x_2y_1-x_3y_2-\ldots-x_1y_n)$ where (x_1, y_1), (x_2, y_2), (x_3, y_3), …, (x_n, y_n) are the coordinates of the vertices, taken in the counterclockwise direction. Again, for convenience sake, students may write the formula in the form Area of triangle $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \cdots & \cdots \\ \cdots & \cdots \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix}$ In real practice, say, to find the area of the quadrilateral with vertices at (1, 3), (4, 1), (4, 4) and (0, 2), teachers should constantly remind students that the order of the points in the formula must be counterclockwise in direction. Therefore, the area of the said polygon is $\frac{1}{2}\begin{vmatrix} 0 & 2 \\ 4 & 1 \\ 4 & 4 \\ 1 & 3 \\ 0 & 2 \end{vmatrix}$ and *NOT* $\frac{1}{2}\begin{vmatrix} 0 & 2 \\ 1 & 3 \\ 4 & 1 \\ 4 & 4 \\ 0 & 2 \end{vmatrix}$ 

Detailed Content	Time Ratio	Notes on Teaching
		A sketch of the points before applying the formula is desirable.
7.3 Inclinations and Slopes of Straight Lines	1*	The slope of a straight line should be defined as the tangent of its angle of inclination to the positive direction of the x-axis, and the range of the inclination is from 0º to 180º. Hence the slope of a line joining the points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$
7.4 Angle between Two Straight Lines	3	Given that the inclinations of two straight lines to the positive x-axis are α and β respectively, the angle between the two straight lines is equal to the difference between α and β. Since $\tan(\alpha-\beta)= [\tan\alpha - \tan\beta] / [1+\tan\alpha\tan\beta]$ $= (m_1-m_2) / (1+m_1m_2)$, the angle between the two straight lines is given by $\tan^{-1}\left\lvert\frac{m_1-m_2}{1+m_1m_2}\right\rvert$ where the absolute signs are included to ensure that the acute angle between the lines is obtained. Now, students should have no problem to see that when two lines with slopes m_1 and m_2 respectively are parallel (i.e. the angle θ between them is zero) if and only if $m_1=m_2$. On the other hand, the two lines are perpendicular (i.e. $\theta=90^0$) if and only if $m_1m_2 = -1$. *Example* The acute angle between the lines x+y=5 and 2x–y=7 is given by $\tan^{-1}\lvert(-1-2)/(1+(-2))\rvert = 71.6^\circ$
7.5 Equations of Straight Lines	1*	Teachers should stress to students that the equation of a straight line is in fact a relationship between the x and y coordinates of any point on the line. The different standard forms are:

Detailed Content	Time Ratio	Notes on Teaching
		(1) The general form Ax+By+C=0 (2) The two-point form $\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$ (3) The point-slope form $y-y_1=m(x-x_1)$ (4) The slope-intercept form y=mx+c (5) The intercept form $x/a+y/b=1$
7.6 The Normal Form	5	Students should be guided to derive the normal form of a straight line in terms of its distance p from the origin ($p \geq 0$) and the inclination α of its normal to the positive x-axis. (Where $0 \leq \alpha < 2\pi$ when p>0, and $0 \leq \alpha < \pi$ when p=0)  With the above diagram, students should have no difficulty to prove that the coordinates (x, y) of any point P on the straight line satisfy the relationship $x\cos\alpha + y\sin\alpha - p=0$, which is the equation of the straight line in normal form. Given the equation of a straight line in its general form Ax+By+C=0, students should be able to convert it to the form $\frac{Ax+By+C}{\pm\sqrt{(A^2+B^2)}}=0$

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		Which is in fact the normal form with the following correspondence $\frac{A}{\pm\sqrt{(A^2+B^2)}}=\cos\alpha$ $\frac{B}{\pm\sqrt{(A^2+B^2)}}=\sin\alpha$ $\frac{C}{\pm\sqrt{(A^2+B^2)}}=-p$ The rules to choose the appropriate sign in the denominator are: (1) If $C \neq 0$, the sign should be opposite to that of C. (2) If C=0, the sign should be the same as that of B. *Example* The line $x-\sqrt{3}y+6=0$ is converted to $\frac{x-\sqrt{3}y+6}{-2}=0$ with p=3 and $\cos\alpha=-\frac{1}{2}$, $\sin\alpha=\sqrt{3}/2$, so $\alpha=120^\circ$. Alternatively, the equation Ax+By+C=0 can be rearranged so that only a positive constant appears on the right-hand side and the whole equation is then divided throughout by $\sqrt{A^2+B^2}$. For example, the above equation is rearranged as $-x+\sqrt{3}y=6$. Thus, $\frac{-x+\sqrt{3}y}{\sqrt{((-1)^2+3)}}=\frac{6}{\sqrt{((-1)^2+3)}}$ or $-\frac{1}{2}x+\frac{\sqrt{3}}{2}y=3$ getting p=3, $\cos\alpha=-\frac{1}{2}$ and $\sin\alpha=\frac{\sqrt{3}}{2}$ as before. The distance from a point to a straight line may be obtained by considering the normal form of a straight line parallel to the given line and passing through the given point (x_0, y_0). The formula $d=\frac{Ax_\circ+By_\circ+C}{\pm\sqrt{(A^2+B^2)}}$ should be memorized.

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		Students should also note that with proper choice of the sign in the denominator, the value obtained bears a sign which tells whether the given point is on the same side of the straight line as the origin or not. To obtain the absolute distance, we take the absolute value of the signed distance. For abler students, elementary treatment of the distance between two parallel lines and the angle bisectors of the lines can be discussed through examples so as to arouse their interest. *Example 1* Find the distance between the parallel lines 12x–5y–10=0 and 12x–5y+16=0. The distance between the two parallel lines may be obtained by taking any point on either of the lines and finding its distance from the other line. Suppose the point(0, –2) on the first line is chosen, the distance of this point from the second line is given by $[(-5)(-2)+16] / \sqrt{(12^2+5^2)} = 2$, and this is the distance between the parallel lines. *Example 2* Find the angle bisectors of the lines x+y–3=0 and x–7y+5=0 If P(x,y) is a point on the angle bisector, then the distance from P to the two lines are equal. Hence we have $\left\lvert \frac{x+y-3}{\sqrt{(1+1)}} \right\rvert = \left\lvert \frac{x-7y+5}{\sqrt{(1+49)}} \right\rvert$ getting x+3y–5=0 or 3x–y–5=0 These are the equations of the angle bisectors of the given lines.
7.7 Family of Straight Lines	3	This topic includes a family of parallel lines with slope m, a family of lines passing through a given point (a, b) and also a family of lines passing through the intersection of the given lines: $A_1x+B_1y+C_1=0$ and $A_2x+B_2y+C_2=0$. Their equations are respectively. y=mx+k y=k(x–a)+b

Detailed Content	Time Ratio	Notes on Teaching
		and $(A_1x+B_1y+C_1) + k(A_2x+B_2y+C_2)=0$ In each case, the equation of the family of straight lines involves a parameter k. N.B. In the case $(A_1x+B_1y+C_1)+k(A_2x+B_2y+C_2)=0$, the line $A_2x+B_2y+C_2=0$ is excluded. So teachers may use the form $\ell(A_1x+B_1y+C_1) + k(A_2x+B_2y+C_2)=0$ where ℓ, k are parameters to include the whole family of lines passing through the intersecting point of $A_1x+B_1y+C_1=0$ and $A_2x+B_2y+C_2=0$. *Example 1* The family of lines parallel to the line 2x–3y+4=0 is given by 2x–3y+h=0, and the family of the lines passing through the point (2, –3) is given by y=k(x–2) –3, where h and k are parameters. *Example 2* Find the equation of the line passing through the intersection of 2x+y–1=0 and 3x–2y–5=0 and at unit distance from the origin. The equation of the required family of lines is (2x+y–1)+k(3x–2y–5)=0 where k is a parameter. Also k should satisfy the condition $\frac{-1-5k}{\sqrt{((2+3k)^2+(1-2k)^2)}} = 1$
7.8 Concept of Locus	1* +2	Teachers should use as many practical examples as possible to illustrate the concept of locus, e.g. paths of a moving point, a moving line, a moving area and moving objects in daily life such as the path of a train running along the rails, the path described by a point on the rim of a rolling cylinder and the trajectory of a football moving in the air. Activities on the geometrical construction of some simple loci may be carried out. Teachers could point out that a straight line is an example of locus in a plane and can be described by the equation of the straight line in the Cartesian plane.

Detailed Content	Time Ratio	Notes on Teaching
7.9 Equation of a Circle, Intersection of a Straight Line and a Circle	7*	*Example* Given two fixed points A(1, 0) and B(0, 3). Find the locus of a point P moving such that its distances from A and B are always equal. The condition PA=PB will lead to a straight line equation. The coordinates of any point on the straight line satisfy the equation of the line and the equation of the line determines the position of the line in the plane. Students should be led to derive the equation of a circle with centre at the origin and radius r from the distance formula. After several examples, they should discover the equations take the form $x^2+y^2=r^2$. Given an equation in the above form, students should recognize that it is a circle with centre at the origin and radius equal to r. Teachers may then consider a more general situation, viz., the centre is not at the origin but with coordinates (h, k). After some practice, students will discover the equations can take one of the following two forms: $x^2+y^2+Dx+Ey+F=0$ or $(x-h)^2+(y-k)^2=r^2$ Given an equation in either of the above forms, students should recognize that it is a circle. They should also know where the centre lies and what the length of its radius is. The equation of a circle passing through three non-linear points should also be discussed. Does a straight line usually cut a circle at two points? Teachers should discuss all the possible cases in relation to the roots of a quadratic equation, especially when the quadratic equation has a double root. The idea of tangency in geometry may be given in terms of the algebraic condition $b^2-4ac=0$. Due amount of examples should be provided and the following show some of them.

Detailed Content	Time Ratio	Notes on Teaching
7.10 Equation of Tangents to a Circle	4	*Example 1* A diameter of a circle has end points at $A(x_1, y_1)$ and $B(x_2, y_2)$. Find the equation of the circle. *Example 2* The vertices of ΔABC are A(–6, 5), B(–3, –4), C(2, 1). Find the equation of the circumscribed circle of ΔABC. *Example 3* Does the line L: $x-y-2=0$ cut the circle C: $x^2+y^2-4x-2y+4=0$? Give reason. Given a circle with equation $x^2+y^2+Dx+Ey+F=0$, students should be guided to derive the equation of the tangent to the circle at a given point (x_1, y_1). The technique of differentiation will be helpful here. Students should be able to obtain the equation $x_1x+y_1y+\frac{1}{2}D(x+x_1)+\frac{1}{2}E(y+y_1)+F=0$. Teachers may then consider a more general situation, viz., the equations of tangents from an external point to the given circle. Students should be shown how this can be done by solving a linear equation of the line passing through the external point and the equation of the circle. Further discussion on finding the equation of tangent by using the fact that the distance from the centre to the tangent is equal to the radius is required. *Example 1* Show that A(1, 4) is a point on the circle C: $x^2+y^2-2x-2y-7=0$. Hence, find the equation of the tangent to C at A. *Example 2* Find the equations of the tangents to the circle $x^2+y^2+6x-2y+5=0$ which are parallel to the line $2x+y+3=0$.

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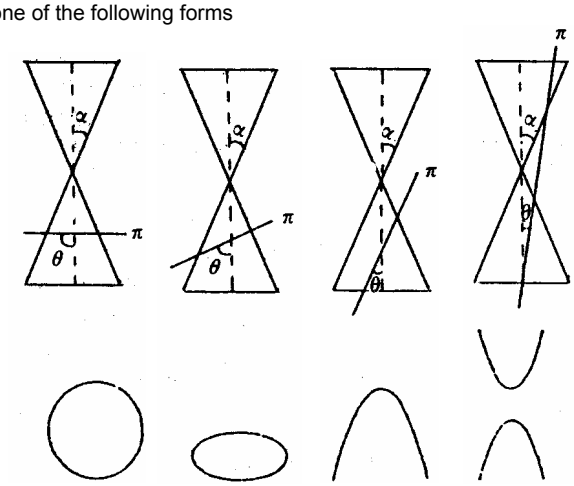
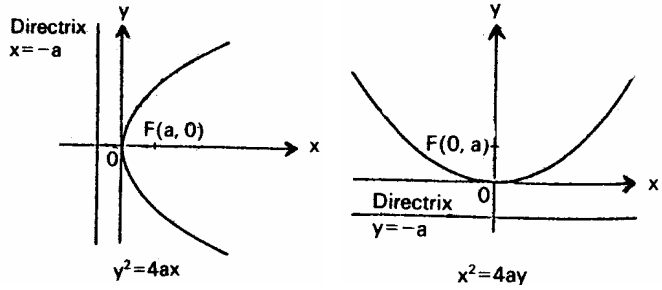
7.11 Family of Circles	4	*Example 3* Two tangents are drawn from the point M(–4, 4) to touch the circle $x^2+y^2-6x-6y-7=0$ at P and Q. Find, (a) the equations of the tangents; (b) $\angle PMQ$; (c) the length of each tangent segment; (d) the area of ΔMPQ. Students should know that the solutions of some kinds of problems are simplified by using the equation of a family of circles. The following 3 cases should be discussed: 1. Family of concentric circles with centres at (a, b). The equation is $(x-a)^2+(y-b)^2=k$ (k>0). 2. Family of circles passing through the points of intersection of the straight line Ax+By+C=0 and the circle $x^2+y^2+Dx+Ey+F=0$. The equation is $x^2+y^2+Dx+Ey+F+k(Ax+By+C)=0$. 3. Family of circles passing through the points of intersection of two given circles with equations respectively equal to $x^2+y^2+D_1x+E_1y+F_1=0$ and $x^2+y^2+D_2x+E_2y+F_2=0$ The equation is $x^2+y^2+D_1x+E_1y+F_1+k(x^2+y^2+D_2x+E_2y+F_2)=0$. where $k \neq -1$. Teachers should explain to students that this equation will reduce to the equation of the common chord of the two circles if k= –1. If, in addition, the two circles touch each other, the common chord will become the common tangent of the circles. Examples such as the following can be introduced. *Example 1* Show that the equation $x^2+y^2-4x+2y+F=0$, where F is a constant, represents a family of concentric circles.

Detailed Content	Time Ratio	Notes on Teaching
7.12 Parabola, Ellipse and Hyperbola, in Standard Position	2	*Example 2* A circle passes through the points of intersection of the line x+y=1 and the circle $x^2+y^2-2x-2y+1=0$. If its centre is on the line 3x–y=3, find its equation. *Example 3* A circle C passes through the point Q(1, 2) and the points of intersection of the circles $x^2+y^2-3x+2y-2=0$ and $x^2+y^2+x+3y-10=0$. Find the equation of C and the equation of the tangent to C at Q. Conic models are very useful teaching aids to introduce the idea of conics. Students should visualize the shapes of the conics and how the conics are formed. A conic is the section of a cone formed when a plane π cuts across the cone. For various angles between π and the axis of the cone, θ the conic takes one of the following forms 

Detailed Content	Time Ratio	Notes on Teaching
		(a) A circle is formed when $\theta = \frac{\pi}{2}$. (b) An ellipse is formed when $\alpha < \theta < \frac{\pi}{2}$. (c) A parabola is formed when $\theta = \alpha$. (d) A hyperbola is formed when $0 \le \theta < \alpha$. Teachers are advised to demonstrate the concept of conics with teaching aids.
7.12.1 Parabola	3	Students are expected to be familiar with the definition of parabola: A parabola is the locus of a point on a plane so that its distances from a fixed point and a fixed distance are equal. Students should be guided to derive the standard equation $y^2=4ax$. Symmetry of the curve and the region of x in which y is defined should be emphasized. The term "focus" should also be introduced. The following diagrams are useful to illustrate the 4 different positions of parabolas, each with the vertex at the origin and the axis on a coordinate axis. 

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		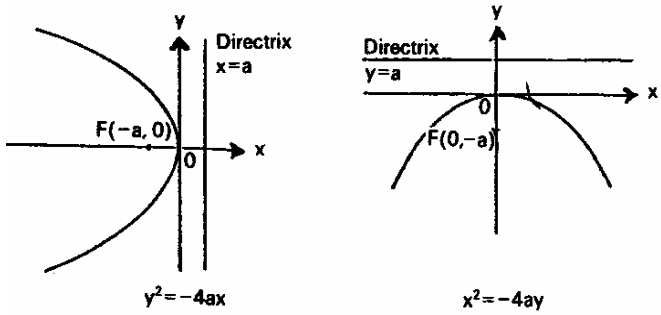 For abler students, simple properties of foci and directrices may be introduced. Examples requiring students to write down the equation of a parabola with a given focus or equation of the directrix and vice versa are necessary. Concrete examples such as the following can be introduced to consolidate students' understanding. *Example 1* Sketch the graph of the parabola $x^2= -6y$. Find also the coordinates of the focus and the equation of the directrix. *Example 2* A parabola passes through the point (–6, 3). If its vertex and axis is the origin and the coordinate axis respectively, find its equation.

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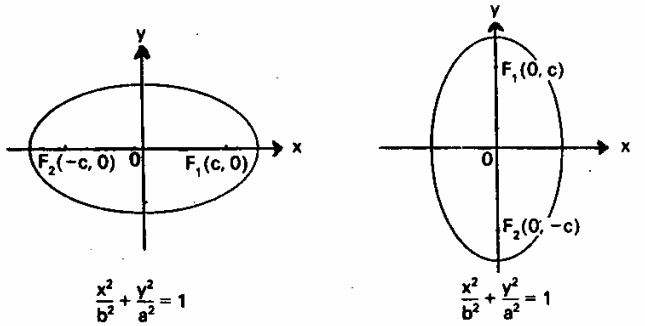
		For abler students, the equations of parabolas in non-standard form such as $y^2=4a(x-h)$ or $x^2=4a(y-k)$ can also be introduced.
7.12.2 Ellipse	3	The standard equation of an ellipse is derived from the definition: It is a locus of a point whose sum of distances from two fixed points is a constant. The terms "focus', "major axis", "semi-major axis", "minor axis" and "semi-minor axis" should be introduced, and the geometric meaning of a, b and c should be stressed. The relation $a^2-b^2=c^2$, where a>b and a>c should also be stated. The various cases with the major axis on either the x-axis or the y-axis and with the centre at the origin can be discussed with the aid of diagrams.  For abler students, simple properties of foci and directrices may be introduced. Students should recognize that in order to obtain the standard equation of an ellipse, it is necessary to know any two of: the length of the major axis, the length of the minor axis and the coordinates of one of the two foci.

Detailed Content	Time Ratio	Notes on Teaching
		Example 1 Find the foci and vertices of the following ellipses and sketch their graphs. (a) $4x^2+9y^2 = 36$ (b) $25x^2+y^2 = 25$ *Example 2* Find the equations of the ellipse with centre at the origin and (a) one focus at (–3, 0), one vertex at (0, 5). (b) the length of the major axis =12, the distance between the foci=6, the foci are on the x-axis. (c) one vertex is (5, 0), and the ellipse passes through (4, 2). *Example 3* Find the equations of the two lines which are both parallel to the line 3x–2y=0 and tangent to the ellipse $4x^2+y^2=16$.
7.12.3 Hyperbola	3	The standard equation of hyperbolas can be derived from its definition: It is the locus of a point whose difference in distances from two fixed points is a constant. Students should acquire the geometrical meanings of a and c. The relation $a^2+b^2=c^2$, where a<c should also be introduced. It is clear from the relation that a is not necessary greater than b. Terms like "focus", "transverse axis", "semi-transverse axis", "conjugate axis" and "semi-conjugate axis" should be explained clearly with the aid of diagram. Students are expected to be familiar with the graphs of hyperbolas, each with centre at the origin and transverse axis on either the x-axis or the y-axis.

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		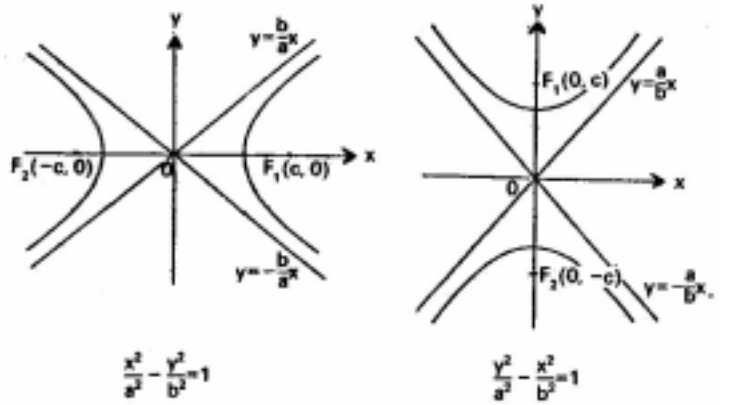 Asymptotes are very useful in sketching hyperbolas. Students should grasp their geometrical meaning. For abler students, simple properties of foci and directrices may be introduced. Students should recognize that in order to obtain the standard equation of an hyperbola, it is necessary to know any two of: the length of the transverse axis, the length of the conjugate axis and the coordinates of one of the two foci. *Example 1* Find the vertices, foci and equations of the asymptotes of the hyperbola $9x^2-y^2 = -9$ and then sketch its graph.

Detailed Content	Time Ratio	Notes on Teaching
		Example 2 The lengths of the transverse axis and the conjugate axis of a hyperbola are respectively equal to 8 and 10. If the centre is at the origin and the foci are on the y-axis, what is its equation? *Example 3* Find the equation of the tangent to the circle $x^2+y^2=9$ which passes through the right focus of the hyperbola $\frac{x^2}{36}-\frac{y^2}{25}=1$. For abler students, rectangular hyperbola can be introduced.
7.13 Simple Parametric Equations and Locus Problems	4	Parametric equation is a way of representation of the equation of a curve. The coordinates of any point (x,y) on the curve are related to a parameter t. Teachers may demonstrate that some parametric equations can be transformed into the equation f(x, y)=0 using the method of elimination. For example, $x=at^2$ and $y=2at$ can be rewritten as $y^2=4ax$. which is the equation of a parabola. Students should recognize the parametric equations of straight lines and conics. Furthermore, teachers should emphasize that a function f(x, y)=0 can have more than one parametric representation. Problems involving loci in parametric forms should be provided to help students master the concept and skill. The following shows some of them. *Example 1* Find an equation connecting x and y for each of the following parametric equations: (*a*) $x=t^2-2t$, $y=t^2+2$. (*b*) $x=\tan\alpha + \cot\alpha$, $y= \tan\alpha - \cot\alpha$

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		Example 2 By putting y=tx, find the parametric equations for the curve $x^3+y^3=6xy$. *Example 3* Given the equation of a parabola $y^2=4ax$. (*a*) Prove that $P_1(at_1^2,2at_1)$ and $P_2(at_2^2,2at_2)$ are points on the parabola. (*b*) If M is the midpoint of P_1P_2, express the coordinates of M in terms of t_1 and t_2. (*c*) If OP_1 is perpendicular to OP_2 where O is the origin, find the equation of the locus of M.
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